

Posterior Cramér-Rao Bounds for Discrete-Time Nonlinear Filtering with Finitely Correlated Noise

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Abstract

In this paper, an approximation recursive formula of the mean-square error lower bound for the discrete-time nonlinear filtering problem when noises of dynamic systems are temporally correlated is derived based on the Van Trees (posterior) version of the Cramér-Rao inequality. The formula is unified in the sense that it can be applied to the multi-step correlated process noise, multi-step correlated measurement noise and multi-step cross-correlated process and measurement noise simultaneously. The lower bound is evaluated by two typical target tracking examples respectively. Both of them show that the new lower bound is significantly different from that of the method which ignores correlation of noises. Thus, when they are applied to sensor selection problems, number of selected sensors becomes very different to obtain a desired estimation performance.

keywords: Nonlinear filtering; correlated noises; posterior Cramér-Rao bounds; target tracking; sensor networks; sensor selection

1 Introduction

The problem of discrete-time nonlinear filtering when noises of dynamic systems are *temporally correlated* (i.e., colored) arises in various applications such as target tracking, navigation, stochastic approximation, adaptive control, robotics, mobile communication [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], just to name a few. For example, in maneuvering target tracking [1], the process noise and target acceleration can be characterized as temporally correlated stochastic process, respectively. In tracking airborne or missile targets using radar data, the measurement noise is significantly correlated when the measurement frequency is high

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[2]. When a system is an airplane and winds are buffeting the plane, an anemometer is used to measure wind speed as an input to Kalman filter. So the random gusts of wind affect both the process (i.e., the airplane dynamics) and the measurement (i.e., the sensed wind speed). Thus, there is a correlation between the process noise and the measurement noise [3]. More detailed results and discussions can be seen in Chapter 7 of the book [3], Chapter 8 of the book [4], and reference therein. As is well known, the optimal estimator for these problems cannot be obtained for *nonlinear* and *non-Gaussian* dynamic systems in general. Besides, assessing the achievable performance of suboptimal filtering techniques may be difficult. A main challenge to researchers in these fields is to find lower bounds corresponding to optimum performance recursively, which give an indication of performance limitations and can be used to determine whether imposed performance requirements are realistic or not.

The most popular lower bound is the well-known Cramér-Rao bound (CRB). In time-invariant statistical models, the estimated parameter vector is usually considered real-valued (non-random). The lower bound is given by the inverse of the Fisher information matrix (see, e.g., [15, 16]). Recently, the authors in [16] discuss the regularity conditions required for the CRB for real-valued (unknown) parameters to hold. It is shown that the commonly assumed requirement that the support of the likelihood function (LF) should be independent of the parameter to be estimated can be replaced by the much weaker requirement that the LF is continuous at the end points of its support. In the time-varying systems context we deal with here, the estimated parameter vector is modeled random. A lower bound that is analogous to the CRB for random parameters was derived in [15]; this bound is also known as the Van Trees version of the CRB, or referred to as posterior CRB (PCRB) [17], where the underlying static random system is assumed to satisfy some regularity conditions which are presented in Section 2. In addition, a general class of Weiss-Weinstein lower bounds in parameter estimation is derived in [18] under less restrictive requirement. The first derivation of a sequential PCRB version applicable to discrete-time dynamic system filtering, the problem addressed in this paper, was done in [19] and then extended in [20, 21, 22, 23]. The most general form of sequential PCRB for discrete-time nonlinear systems was presented in [17], [24]. Together with the original static form of the CRB, these results served as a basis for a large number of applications [25, 26, 27, 28, 29, 30, 31].

In this paper, we focus on an approximation recursive derivation of the PCRB for the discrete-time nonlinear and non-Gaussian filtering problem when noises of dynamic systems are temporally correlated. The derived formula is unified in the sense that it can be applied to the multi-step correlated process noise, multi-step correlated measurement noise and multi-step cross-correlated process and measurement noise simultaneously. The derivation differs from the other approaches that instead consider the three cases separately and assume the linear or Gaussian dynamic systems. Although the unified formula can come across the three cases of finite-step correlated noises, a few corollaries with simpler formulae follow to elucidate special cases, which may be used more frequently. The main results are presented in Section 3. In Section IV, the new lower bound is evaluated by two typical target tracking examples, respectively. Both of them show that the new lower bound is significantly different from that of the existing methods. Thus, when they are applied to sensor selection problems, simulations show that the new formula can derive a more accurate number of

selected sensors to obtain a desired estimation performance. Conclusions are drawn in Section 5. In order to enhance readability, all proofs are given in Appendices.

2 Preliminaries

2.1 Problem Formulation

Consider a nonlinear dynamic system

$$\mathbf{x}_{k+1} = f_k(\mathbf{x}_k, \mathbf{w}_k) \quad (1)$$

$$\mathbf{z}_k = h_k(\mathbf{x}_k, \mathbf{v}_k) \quad (2)$$

where $\mathbf{x}_k \in \mathbb{R}^r$ is the state to be estimated at time k , r is the dimension of the state; $\mathbf{z}_k \in \mathbb{R}^n$ is the measurement vector. The function f_k and h_k are nonlinear functions in general. $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ are noises both temporally *finite-step correlated*, respectively. We discuss the following three cases:

1. The process noises are *l-step auto-correlated* if their probability density functions satisfy $p(\mathbf{w}_k, \mathbf{w}_{k-i}) \neq p(\mathbf{w}_k)p(\mathbf{w}_{k-i})$, $p(\mathbf{w}_k, \mathbf{w}_{k-j}) = p(\mathbf{w}_k)p(\mathbf{w}_{k-j})$, for $i = 1, \dots, l, j = l+1, \dots, k, k \geq l+1$. We denote by 0-step correlated process noise if they are temporally independent.
2. The measurement noises are *l-step auto-correlated* if $p(\mathbf{v}_k, \mathbf{v}_{k-i}) \neq p(\mathbf{v}_k)p(\mathbf{v}_{k-i})$, and $p(\mathbf{v}_k, \mathbf{v}_{k-j}) = p(\mathbf{v}_k)p(\mathbf{v}_{k-j})$, for $i = 1, \dots, l, j = l+1, \dots, k, k \geq l+1$. We denote by 0-step correlated measurement noise if they are temporally independent.
3. The measurement noise is *backward l-step cross-correlated* with process noise, if $p(\mathbf{v}_k, \mathbf{w}_{k-i}) \neq p(\mathbf{v}_k)p(\mathbf{w}_{k-i})$, and $p(\mathbf{v}_k, \mathbf{w}_{k-j}) = p(\mathbf{v}_k)p(\mathbf{w}_{k-j})$, for $i = 1, \dots, l, j = l+1, \dots, k, k \geq l+1$; The measurement noise is *forward l-step cross-correlated* with process noise, if $p(\mathbf{v}_k, \mathbf{w}_{k-1+i}) \neq p(\mathbf{v}_k)p(\mathbf{w}_{k-1+i})$, and $p(\mathbf{v}_k, \mathbf{w}_{k-1+j}) = p(\mathbf{v}_k)p(\mathbf{w}_{k-1+j})$, for $i = 1, \dots, l, j = l+1, \dots, k, k \geq l+1$; We denote that the measurement noise is forward and backward 0-step correlated with the process noise if they are mutually independent.

Note that if the measurement noise is finite-step correlated to the process noise, then the process noise is also finite-step correlated to the measurement noise. For example, the observation noises at times $k-1$ and k are correlated to process noise at time $k-2$, then process noises at times $k-2$ and $k-1$ are correlated to observation noise at time k . Thus, their recursive formulae are the same and we only consider the former.

Since, in target tracking, the three correlated cases may be encountered simultaneously (see, e.g., [1, 2]) and the optimal estimator for these problems cannot be obtained for *nonlinear* and *non-Gaussian* dynamic systems in general, the goal of this paper is to derive a unified lower bound recursively, which can be used to determine whether imposed performance requirements are realistic or not.

2.2 Posterior Cramér-Rao Bounds

Let \mathbf{x} be a r -dimensional random parameter vector and \mathbf{z} be a measurement vector, let $p_{\mathbf{x},\mathbf{z}}(\mathbf{X}, \mathbf{Z})$ be a joint density of the pair (\mathbf{x}, \mathbf{z}) . The mean-square error of any estimate $\hat{\mathbf{x}}(\mathbf{Z})$ of \mathbf{x} satisfies the inequality

$$\mathbf{P} \triangleq E\{[\hat{\mathbf{x}}(\mathbf{Z}) - \mathbf{x}][\hat{\mathbf{x}}(\mathbf{Z}) - \mathbf{x}]^T\} \geq \mathbf{J}^{-1} \quad (3)$$

where \mathbf{J} is the $r \times r$ (Fisher) information matrix with the elements

$$\mathbf{J}_{ij} = E\left[-\frac{\partial^2 \ln p_{\mathbf{x},\mathbf{z}}(\mathbf{X}, \mathbf{Z})}{\partial \mathbf{X}_i \partial \mathbf{X}_j}\right] \quad i, j = 1, \dots, r \quad (4)$$

and the expectation is over both \mathbf{x} and \mathbf{z} . The superscript “ T ” in (3) denotes the transpose of a matrix. The following conditions are assumed to exist:

1. $\frac{\partial \ln p_{\mathbf{x},\mathbf{z}}(\mathbf{X}, \mathbf{Z})}{\partial \mathbf{X}}$ is absolutely integrable with respect to \mathbf{X} and \mathbf{Z} .
2. $\frac{\partial^2 \ln p_{\mathbf{x},\mathbf{z}}(\mathbf{X}, \mathbf{Z})}{\partial \mathbf{X}^2}$ is absolutely integrable with respect to \mathbf{X} and \mathbf{Z} .
3. The conditional expectation of the error, given \mathbf{X} , is

$$B(\mathbf{X}) = \int_{-\infty}^{+\infty} [\hat{\mathbf{x}}(\mathbf{Z}) - \mathbf{X}] p_{\mathbf{z}|\mathbf{x}}(\mathbf{Z}|\mathbf{X}) d\mathbf{Z}$$

and assume that

$$\begin{aligned} \lim_{\mathbf{X}_i \rightarrow \infty} B(\mathbf{X}) p_{\mathbf{x}}(\mathbf{X}) &= 0, \quad \lim_{\mathbf{X}_i \rightarrow -\infty} B(\mathbf{X}) p_{\mathbf{x}}(\mathbf{X}) = 0, \\ \text{for } i &= 1, \dots, r. \end{aligned}$$

The proof is given in [15].

Assume now that the parameter \mathbf{x} is decomposed into two parts as $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T]^T$, and the information matrix \mathbf{J} is correspondingly decomposed into blocks

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{12}^T & \mathbf{J}_{22} \end{pmatrix} \quad (5)$$

It can easily be shown that the covariance of estimation of \mathbf{x}_2 is lower bounded by the right-lower block of \mathbf{J}^{-1} , i.e.,

$$\begin{aligned} \mathbf{P}_2 &\triangleq E\{[\hat{\mathbf{x}}_2(\mathbf{Z}) - \mathbf{x}_2][\hat{\mathbf{x}}_2(\mathbf{Z}) - \mathbf{x}_2]^T\} \\ &\geq [\mathbf{J}_{22} - \mathbf{J}_{12}^T \mathbf{J}_{11}^{-1} \mathbf{J}_{12}]^{-1} \end{aligned} \quad (6)$$

assuming that \mathbf{J}_{11}^{-1} exists. The matrix $\mathbf{J}_{22} - \mathbf{J}_{12}^T \mathbf{J}_{11}^{-1} \mathbf{J}_{12}$ is called the information submatrix for parameter \mathbf{x}_2 in [17].

Note that the joint probability densities $p(\mathbf{X}_k, \mathbf{Z}_k)$ of $\mathbf{X}_k = (\mathbf{x}_0^T, \dots, \mathbf{x}_k^T)^T$ and $\mathbf{Z}_k = (\mathbf{z}_0^T, \dots, \mathbf{z}_k^T)^T$ for an arbitrary k is determined by Equations (1) and (2) together with $p(\mathbf{x}_0)$ and also by the noise pdfs [19]. The conditional probability densities $p(\mathbf{x}_{k+1}|\mathbf{X}_k, \mathbf{Z}_k)$ and $p(\mathbf{z}_{k+1}|\mathbf{X}_{k+1}, \mathbf{Z}_k)$ can be obtained from (1) and (2), respectively, under suitable hypotheses. In this paper, $p(\mathbf{X}_k, \mathbf{Z}_k)$ is denoted by p_k for brevity. From a Bayesian perspective, the joint probability function of \mathbf{X}_{k+1} and \mathbf{Z}_{k+1} can be written as

$$p_{k+1} = p(\mathbf{X}_{k+1}, \mathbf{Z}_{k+1}) \quad (7)$$

$$= p_k p(\mathbf{x}_{k+1}|\mathbf{X}_k, \mathbf{Z}_k) p(\mathbf{z}_{k+1}|\mathbf{X}_{k+1}, \mathbf{Z}_k). \quad (8)$$

In addition, define ∇ and \triangle be the first and second-order operator partial derivatives, respectively

$$\nabla_\alpha = \left(\frac{\partial}{\partial \alpha_1} \quad \frac{\partial}{\partial \alpha_2} \quad \dots \quad \frac{\partial}{\partial \alpha_n} \right)^T \quad \forall \alpha \in R^n, \quad (9)$$

$$\triangle_\alpha^\beta = \nabla_\alpha \nabla_\beta^T. \quad (10)$$

Using this notation and (7), (4) can be written as

$$\mathbf{J}(\mathbf{X}_k) = E \left(-\triangle_{\mathbf{X}_k}^{\mathbf{X}_k} \ln p(\mathbf{X}_k, \mathbf{Z}_k) \right) = E \left(-\triangle_{\mathbf{X}_k}^{\mathbf{X}_k} \ln p_k \right). \quad (11)$$

Decompose state vector \mathbf{X}_k as $\mathbf{X}_k = (\mathbf{X}_{k-1}^T, \mathbf{x}_k^T)^T$ and the $(kr \times kr)$ information matrix $\mathbf{J}(\mathbf{X}_k)$ correspondingly as

$$\mathbf{J}(\mathbf{X}_k) = \begin{pmatrix} \mathbf{A}_k^{11} & \mathbf{A}_k^{12} \\ \mathbf{A}_k^{21} & \mathbf{A}_k^{22} \end{pmatrix} \quad (12)$$

where

$$\mathbf{A}_k^{11} = E(-\triangle_{\mathbf{X}_{k-1}}^{\mathbf{X}_{k-1}} \ln p_k),$$

$$\mathbf{A}_k^{12} = E(-\triangle_{\mathbf{X}_{k-1}}^{\mathbf{x}_k} \ln p_k),$$

$$\mathbf{A}_k^{21} = E(-\triangle_{\mathbf{x}_k}^{\mathbf{X}_{k-1}} \ln p_k),$$

$$\mathbf{A}_k^{22} = E(-\triangle_{\mathbf{x}_k}^{\mathbf{x}_k} \ln p_k).$$

Thus, the posterior information submatrix for estimating \mathbf{x}_k , denoted by \mathbf{J}_k , which is given as the inverse of the $(r \times r)$ right block of $[\mathbf{J}(\mathbf{X}_k)]^{-1}$, i.e.,

$$\mathbf{J}_k = \mathbf{A}_k^{22} - \mathbf{A}_k^{21} \mathbf{A}_k^{11-1} \mathbf{A}_k^{12}. \quad (13)$$

\mathbf{J}_k^{-1} is the PCRB of estimating state vector \mathbf{x}_k .

In the following, we derive the recursive formula of the posterior information submatrices $\{\mathbf{J}_k\}$ when the noises of dynamic systems are finite-step correlated.

3 Main results

In this section, we address the recursive formula of the posterior information submatrices $\{\mathbf{J}_k\}$ when the noises of dynamic systems are finite-step correlated. Let us give some remarks about the general matrix M_k on the notation:

1. $\mathbf{M}_k^{i,j}$, $i, j = 1, \dots, l$ denotes the i -th row and j -th column block of the $(l \times l)$ block matrix \mathbf{M}_k at time k ;
2. If $i \leq 0$, or $j \leq 0$, then $\mathbf{M}_k^{i,j} = 0$;
3. If $i > l$, or $j > l$, then $\mathbf{M}_k^{i,j} = 0$.

When the measurement noise and the process noise of the dynamic system (1)-(2) are temporally auto-correlated and cross-correlated simultaneously, we have the following unified recursion as follows.

Proposition 3.1. *If the measurement noise is l_1 -step auto-correlated ($l_1 \geq 0$), the process noise is l_2 -step auto-correlated ($l_2 \geq 0$), and the measurement noise is backward l_3 -step and forward l_4 -step cross-correlated with the process noise ($l_3 \geq 0$, $l_4 \geq 0$), then the sequence $\{\mathbf{J}_k\}$ of posterior information submatrices for estimating state vector $\{\mathbf{x}_k\}$ approximately obeys the recursion*

$$\mathbf{J}_{k+1} = \mathbf{D}_k^{22} - \mathbf{D}_k^{21}(\mathbf{D}_k^{11} + \mathbf{E}_k)^{-1}\mathbf{D}_k^{12}, \quad (14)$$

where the recursive terms \mathbf{E}_k , \mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} are calculated as the following three cases. In order to facilitate the discussion, with a slight abuse of notations, we denote by $l'_3 \triangleq \max\{l_3, 1\}$, $l'_2 \triangleq \max\{l_2, 1\}$.

1. $l'_3 > l'_2 + 1$:

The i -th row and j -th column block of the matrix \mathbf{E}_k are recursively calculated as

$$\begin{aligned} \mathbf{E}_k^{i,j} &= \mathbf{E}_{k-1}^{i+1,j+1} + \mathbf{B}_{k-1}^{i+l'_2-l'_3+2,j+l'_2-l'_3+2} \\ &\quad + \mathbf{C}_{k-1}^{i+1,j+1} - (\mathbf{E}_{k-1}^{i+1,1} + \mathbf{C}_{k-1}^{i+1,1}) \\ &\quad \cdot (\mathbf{E}_{k-1}^{1,1} + \mathbf{C}_{k-1}^{1,1})^{-1} (\mathbf{E}_{k-1}^{1,j+1} + \mathbf{C}_{k-1}^{1,j+1}) \end{aligned} \quad (15)$$

for $i = 1, 2, \dots, l'_3 - 1$, $j = 1, 2, \dots, l'_3 - 1$

where

$$\mathbf{B}_k^{i,j} = \begin{cases} E(-\Delta_{\mathbf{x}_{k-l'_2+i}}^{\mathbf{x}_{k-l'_2+j}} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \dots, \mathbf{x}_{k-l'_2+1})) \\ \text{for } i, j = 1, 2, \dots, l'_2 + 1, l_4 = 0, \\ \\ E(-\Delta_{\mathbf{x}_{k-l'_2+i}}^{\mathbf{x}_{k-l'_2+j}} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \dots, \mathbf{x}_{k-l'_2+1}, \\ \mathbf{z}_k, \dots, \mathbf{z}_{k-l_4+1})) \\ \text{for } i, j = 1, 2, \dots, l'_2 + 1, l_4 > 0, \end{cases} \quad (16)$$

$$\mathbf{C}_k^{i,j} = \begin{cases} E(-\Delta_{\mathbf{x}_{k+1+i-l'_3}}^{\mathbf{x}_{k+1+j-l'_3}} \ln p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \dots, \mathbf{x}_{k-l'_3+2})) \\ \text{for } i, j = 1, 2, \dots, l'_3, l_1 = 0 \\ \\ E(-\Delta_{\mathbf{x}_{k+1+i-l'_3}}^{\mathbf{x}_{k+1+j-l'_3}} \ln p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \dots, \mathbf{x}_{k-l'_3+2}, \\ \mathbf{z}_k, \dots, \mathbf{z}_{k-l_1+1})) \\ \text{for } i, j = 1, 2, \dots, l'_3, l_1 > 0. \end{cases} \quad (17)$$

\mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} in (14) can be calculated as follows:

$$\mathbf{D}_k^{22} = \mathbf{C}_k^{l'_3, l'_3} + \mathbf{B}_k^{l'_2+1, l'_2+1}, \quad (18)$$

$$\mathbf{D}_k^{21} = \begin{pmatrix} \mathbf{C}_k^{l'_3 1} & \dots & \mathbf{C}_k^{l'_3, l'_3-l'_2-1} & \dots & \mathbf{C}_k^{l'_3, l'_3-1} \\ & & +\mathbf{B}_k^{l'_2+1, 1} & & +\mathbf{B}_k^{l'_2+1, l'_2} \end{pmatrix} = (\mathbf{D}_k^{12})^T, \quad (19)$$

$$\mathbf{D}_k^{11} = \begin{pmatrix} \mathbf{C}_k^{1,1} & \dots & \mathbf{C}_k^{1, l'_3-l'_2} & \dots & \mathbf{C}_k^{1, l'_3-1} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \mathbf{C}_k^{l'_3-l'_2, 1} & \dots & \mathbf{C}_k^{l'_3-l'_2, l'_3-l'_2} & \dots & \mathbf{C}_k^{l'_3-l'_2, l'_3-1} \\ & & +\mathbf{B}_k^{1,1} & & +\mathbf{B}_k^{1, l'_2} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \mathbf{C}_k^{l'_3-1, 1} & \dots & \mathbf{C}_k^{l'_3-1, l'_3-l'_2} & \dots & \mathbf{C}_k^{l'_3-1, l'_3-1} \\ & & +\mathbf{B}_k^{l'_2, 1} & & +\mathbf{B}_k^{l'_2, l'_2} \end{pmatrix}. \quad (20)$$

2. $l'_3 < l'_2 + 1$

The i -th row and j -th column block of the matrix \mathbf{E}_k are recursively calculated as

$$\begin{aligned}\mathbf{E}_k^{i,j} &= \mathbf{E}_{k-1}^{i+1,j+1} + \mathbf{C}_{k-1}^{i+l'_3-l'_2,j+l'_3-l'_2} + \mathbf{B}_{k-1}^{i+1,j+1} \\ &\quad - (\mathbf{E}_{k-1}^{i+1,1} + \mathbf{B}_{k-1}^{i+1,1})(\mathbf{E}_{k-1}^{1,1} + \mathbf{B}_{k-1}^{1,1})^{-1} \\ &\quad \cdot (\mathbf{E}_{k-1}^{1,j+1} + \mathbf{B}_{k-1}^{1,j+1})\end{aligned}\quad (21)$$

for $i = 1, 2, \dots, l'_2$, $j = 1, 2, \dots, l'_2$,

where $\mathbf{C}_k^{i,j}$ and $\mathbf{B}_k^{i,j}$ are defined in (16)-(17). \mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} in (14) can be calculated as follows:

$$\mathbf{D}_k^{22} = \mathbf{C}_k^{l'_3, l'_3} + \mathbf{B}_k^{l'_2+1, l'_2+1}, \quad (22)$$

$$\mathbf{D}_k^{21} = \begin{pmatrix} \mathbf{B}_k^{l'_2+1, 1} & \dots & \mathbf{B}_k^{l'_2+1, l'_2-l'_3} & \dots & \mathbf{B}_k^{l'_2+1, l'_2} \\ & & + \mathbf{C}_k^{l'_3, 1} & & + \mathbf{C}_k^{l'_3, l'_3-1} \end{pmatrix} = (\mathbf{D}_k^{12})^T, \quad (23)$$

$$\mathbf{D}_k^{11} = \begin{pmatrix} \mathbf{B}_k^{1,1} & \dots & \mathbf{B}_k^{1, l'_2} \\ \vdots & \ddots & \vdots \\ \mathbf{B}_k^{l'_2, 1} & \dots & \mathbf{B}_k^{l'_2, l'_2} + \mathbf{C}_k^{l'_3-1, l'_3-1} \end{pmatrix}. \quad (24)$$

3. $l'_3 = l'_2 + 1$

The i -th row and j -th column block of the matrix \mathbf{E}_k are recursively calculated as

$$\begin{aligned}\mathbf{E}_k^{i,j} &= \mathbf{E}_{k-1}^{i+1,j+1} + \mathbf{C}_{k-1}^{i+1,j+1} + \mathbf{B}_{k-1}^{i+1,j+1} \\ &\quad - (\mathbf{E}_{k-1}^{i+1,1} + \mathbf{B}_{k-1}^{i+1,1} + \mathbf{C}_{k-1}^{i+1,1}) \\ &\quad \cdot (\mathbf{E}_{k-1}^{1,1} + \mathbf{B}_{k-1}^{1,1} + \mathbf{C}_{k-1}^{1,1})^{-1} \\ &\quad \cdot (\mathbf{E}_{k-1}^{1,j+1} + \mathbf{B}_{k-1}^{1,j+1} + \mathbf{C}_{k-1}^{1,j+1})\end{aligned}\quad (25)$$

for $i = 1, 2, \dots, l'_3 - 1$, $j = 1, 2, \dots, l'_3 - 1$,

where $\mathbf{C}_k^{i,j}$ and $\mathbf{B}_k^{i,j}$ are defined in (16)-(17). \mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} in (14) can be calculated as follows:

$$\mathbf{D}_k^{22} = \mathbf{C}_k^{l'_3, l'_3} + \mathbf{B}_k^{l'_3, l'_3}, \quad (26)$$

$$\begin{aligned} \mathbf{D}_k^{21} &= \begin{pmatrix} \mathbf{B}_k^{l'_3, 1} & \dots & \mathbf{B}_k^{l'_3, l'_3-1} \\ +\mathbf{C}_k^{l'_3, 1} & & +\mathbf{C}_k^{l'_3, l'_3-1} \end{pmatrix} \\ &= (\mathbf{D}_k^{12})^T, \end{aligned} \quad (27)$$

$$\mathbf{D}_k^{11} = \begin{pmatrix} \mathbf{B}_k^{1,1} & \dots & \mathbf{B}_k^{1, l'_3-1} \\ +\mathbf{C}_k^{1,1} & & +\mathbf{C}_k^{1, l'_3-1} \\ \dots & \ddots & \dots \\ \mathbf{B}_k^{l'_3-1, 1} & \dots & \mathbf{B}_k^{l'_3-1, l'_3-1} \\ +\mathbf{C}_k^{l'_3-1, 1} & & +\mathbf{C}_k^{l'_3-1, l'_3-1} \end{pmatrix}. \quad (28)$$

Proof: See the Appendix.

Remark 3.2. The difficulty of the recursion is the derivation of the recursive matrix \mathbf{E}_k , which thanks to two lemmas about the inverse of a matrix given in Appendix and Schur complement. Although the derivation is very complicated, the final formula is not complicated. The finite-step correlation of noises is used to determine (16) and (17), which can be calculated by analytical or numerical methods. Note that both of them are approximation equations (see proof in the appendix). The initial information submatrix \mathbf{E}_0^{ij} can be calculated from the a priori probability function $p(\mathbf{X}_{l_{max}}, \mathbf{Z}_{l_{max}})$ where $l_{max} = \max\{l_1, l_2, l_3, l_4\}$.

Although the unified formula can come across the three cases of finite-step correlated noises, a few corollaries with simpler formulae follow to elucidate special cases, which may be used more frequently.

Corollary 3.3. *If the measurement noise is backward l -step cross-correlated with the process noise ($l \geq 0$), the process noise and measurement noise are temporally independent, respectively, i.e., $l_1 = 0$, $l_2 = 0$, $l_3 = l$, $l_4 = 0$, then the sequence $\{\mathbf{J}_k\}$ of posterior information submatrices for estimating state vector $\{\mathbf{x}_k\}$ obeys the recursion*

$$\mathbf{J}_{k+1} = \mathbf{D}_k^{22} - \mathbf{D}_k^{21}(\mathbf{D}_k^{11} + \mathbf{E}_k)^{-1}\mathbf{D}_k^{12}, \quad (29)$$

where the recursive term \mathbf{E}_k is calculated as follows:

1. If $l = 1$ or $l = 0$, then

$$\begin{aligned} \mathbf{E}_k &= \mathbf{B}_{k-1}^{2,2} + \mathbf{C}_{k-1}^{1,1} \\ &\quad - \mathbf{B}_{k-1}^{2,1}(\mathbf{E}_{k-1} + \mathbf{B}_{k-1}^{1,1})^{-1}\mathbf{B}_{k-1}^{1,2} \end{aligned} \quad (30)$$

where

$$\begin{aligned}
\mathbf{B}_{k-1}^{1,1} &= E(-\Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_{k-1}} \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})), \\
\mathbf{B}_{k-1}^{1,2} &= E(-\Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_k} \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})), \\
\mathbf{B}_{k-1}^{2,1} &= E(-\Delta_{\mathbf{x}_k}^{\mathbf{x}_{k-1}} \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})), \\
\mathbf{C}_{k-1}^{1,1} &= E(-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \ln p(\mathbf{z}_k|\mathbf{x}_k)).
\end{aligned}$$

\mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} in (29) are calculated as follows

$$\begin{aligned}
\mathbf{D}_k^{22} &= \mathbf{B}_k^{2,2} + \mathbf{C}_k^{1,1}, \\
\mathbf{D}_k^{21} &= \mathbf{B}_k^{2,1} = (\mathbf{D}_k^{12})^T, \\
\mathbf{D}_k^{11} &= \mathbf{B}_k^{1,1}.
\end{aligned}$$

2. If $l = 2$, then

$$\begin{aligned}
\mathbf{E}_k &= \mathbf{C}_{k-1}^{2,2} + \mathbf{B}_{k-1}^{2,2} - (\mathbf{B}_{k-1}^{2,1} + \mathbf{C}_{k-1}^{2,1}) \\
&\quad \cdot (\mathbf{E}_{k-1} + \mathbf{B}_{k-1}^{1,1} + \mathbf{C}_{k-1}^{1,1})^{-1} (\mathbf{C}_{k-1}^{1,2} + \mathbf{B}_{k-1}^{1,2}),
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\mathbf{B}_{k-1}^{m,n} &= E(-\Delta_{\mathbf{x}_{k-2+m}}^{\mathbf{x}_{k-2+n}} \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})) \\
&\quad \text{for } m, n = 1, 2, \\
\mathbf{C}_{k-1}^{m,n} &= E(-\Delta_{\mathbf{x}_{k-2+m}}^{\mathbf{x}_{k-2+n}} \ln p(\mathbf{z}_k|\mathbf{x}_k, \mathbf{x}_{k-1})) \\
&\quad \text{for } m, n = 1, 2.
\end{aligned}$$

\mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} in (29) are calculated as follows

$$\begin{aligned}
\mathbf{D}_k^{22} &= \mathbf{C}_k^{2,2} + \mathbf{B}_k^{2,2}, \\
\mathbf{D}_k^{21} &= \mathbf{B}_k^{2,1} + \mathbf{C}_k^{2,1} = (\mathbf{D}_k^{12})^T, \\
\mathbf{D}_k^{11} &= \mathbf{B}_k^{1,1} + \mathbf{C}_k^{1,1}.
\end{aligned}$$

3. If $l > 2$, then the i -th row and j -th column block of the matrix \mathbf{E}_k is recursively calculated as

$$\begin{aligned}
\mathbf{E}_k^{i,j} &= \mathbf{E}_{k-1}^{i+1,j+1} + \mathbf{B}_{k-1}^{i+3-l,j+3-l} + \mathbf{C}_{k-1}^{i+1,j+1} - \\
&\quad (\mathbf{E}_{k-1}^{i+1,1} + \mathbf{C}_{k-1}^{i+1,1})(\mathbf{E}_{k-1}^{1,1} + \mathbf{C}_{k-1}^{1,1})^{-1} \\
&\quad \cdot (\mathbf{E}_{k-1}^{1,j+1} + \mathbf{C}_{k-1}^{1,j+1}),
\end{aligned} \tag{32}$$

for $i = 1, 2, \dots, l'_3 - 1$, $j = 1, 2, \dots, l'_3 - 1$

where

$$\mathbf{C}_{k-1}^{m,n} = E(-\Delta_{\mathbf{x}_{k+m-l}}^{\mathbf{x}_{k+n-l}} \ln p(\mathbf{z}_k | \mathbf{x}_k, \dots, \mathbf{x}_{k+1-l}))$$

for $m, n = 1, 2, \dots, l$.

$$\begin{aligned} \mathbf{B}_{k-1}^{1,1} &= E(-\Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_{k-1}} \ln p(\mathbf{x}_k | \mathbf{x}_{k-1})), \\ \mathbf{B}_{k-1}^{1,2} &= E(-\Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_k} \ln p(\mathbf{x}_k | \mathbf{x}_{k-1})) = (\mathbf{B}_k^{2,1})^T, \\ \mathbf{B}_{k-1}^{2,2} &= E(-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \ln p(\mathbf{x}_k | \mathbf{x}_{k-1})). \end{aligned}$$

\mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} in (29) are calculated as follows

$$\begin{aligned} \mathbf{D}_k^{22} &= \mathbf{C}_k^{l,l} + \mathbf{B}_k^{2,2}, \\ \mathbf{D}_k^{21} &= \begin{pmatrix} \mathbf{C}_k^{l,1} & \dots & \mathbf{B}_k^{2,1} + \mathbf{C}_k^{l,l-1} \end{pmatrix} = (\mathbf{D}_k^{12})^T, \\ \mathbf{D}_k^{11} &= \begin{pmatrix} \mathbf{C}_k^{1,1} & \dots & \mathbf{C}_k^{1,l-1} \\ \dots & \ddots & \dots \\ \mathbf{C}_k^{l-1,1} & \dots & \mathbf{B}_k^{1,1} + \mathbf{C}_k^{l-1,l-1} \end{pmatrix}. \end{aligned}$$

Proof: See the Appendix.

Corollary 3.4. *If the process noise is l -step auto-correlated ($l \geq 0$), the measurement noise is temporally independent, and the process noise and the measurement noise are mutually independent, i.e., $l_1 = 0$, $l_2 = l$, $l_3 = 0$, $l_4 = 0$, then the sequence $\{\mathbf{J}_k\}$ of posterior information submatrices for estimating state vector $\{\mathbf{x}_k\}$ obeys the recursion*

$$\mathbf{J}_{k+1} = \mathbf{D}_k^{22} - \mathbf{D}_k^{21}(\mathbf{E}_k + \mathbf{D}_k^{11})^{-1}\mathbf{D}_k^{12} \quad (33)$$

where the i -th row and j -th column block of the matrix \mathbf{E}_k is calculated as follows:

$$\begin{aligned} \mathbf{E}_k^{i,j} &= \mathbf{E}_{k-1}^{i+1,j+1} + \mathbf{C}_{k-1}^{i+1-l'_2, j+1-l'_2} + \mathbf{B}_{k-1}^{i+1,j+1} \\ &\quad - (\mathbf{E}_{k-1}^{i+1,1} + \mathbf{B}_{k-1}^{i+1,1})(\mathbf{E}_{k-1}^{1,1} + \mathbf{B}_{k-1}^{1,1})^{-1} \\ &\quad \cdot (\mathbf{E}_{k-1}^{1,j+1} + \mathbf{B}_{k-1}^{1,j+1}) \end{aligned} \quad (34)$$

for $i = 1, 2, \dots, l'_2$, $j = 1, 2, \dots, l'_2$

where

$$\mathbf{B}_{k-1}^{i,j} = E(-\Delta_{\mathbf{x}_{k+i-l'_2-1}}^{\mathbf{x}_{k+j-l'_2-1}} \ln p(\mathbf{x}_k | \mathbf{x}_{k-1}, \dots, \mathbf{x}_{k-l'_2})) \quad (35)$$

$$\mathbf{C}_{k-1}^{1,1} = E(-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \ln p(\mathbf{z}_k|\mathbf{x}_k)), \quad (36)$$

$$\text{for } i = 1, 2, \dots, l'_2 + 1, j = 1, 2, \dots, l'_2 + 1.$$

$\mathbf{D}_k^{11}, \mathbf{D}_k^{12}, \mathbf{D}_k^{21}$ and \mathbf{D}_k^{22} in (33) are calculated as follows

$$\mathbf{D}_k^{22} = \mathbf{B}_k^{l'_2+1, l'_2+1} + \mathbf{C}_k^{1,1}, \quad (37)$$

$$\mathbf{D}_k^{21} = \begin{pmatrix} \mathbf{B}_k^{l'_2+1, 1} & \dots & \mathbf{B}_k^{l'_2+1, l'_2} \end{pmatrix} = (\mathbf{D}_k^{12})^T, \quad (38)$$

$$\mathbf{D}_k^{11} = \begin{pmatrix} \mathbf{B}_k^{1,1} & \dots & \mathbf{B}_k^{1, l'_2} \\ \vdots & \ddots & \vdots \\ \mathbf{B}_k^{l'_2, 1} & \dots & \mathbf{B}_k^{l'_2, l'_2} \end{pmatrix}. \quad (39)$$

Proof: See the Appendix.

Corollary 3.5. *If the measurement noise is l -step auto-correlated ($l \geq 0$), the process noise is temporally independent, and the process noise and the measurement noise are mutually independent, i.e., $l_1 = l, l_2 = 0, l_3 = 0, l_4 = 0$, then the sequence $\{\mathbf{J}_k\}$ of posterior information submatrices for estimating state vector $\{\mathbf{x}_k\}$ obeys the recursion*

$$\mathbf{J}_{k+1} = \mathbf{D}_k^{22} - \mathbf{D}_k^{21}(\mathbf{D}_k^{11} + \mathbf{J}_k)^{-1}\mathbf{D}_k^{12} \quad (40)$$

where

$$\begin{aligned} \mathbf{D}_k^{11} &= E(-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k)), \\ \mathbf{D}_k^{21} &= E(-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_k} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k)) = (\mathbf{D}_k^{12})^T, \\ \mathbf{D}_k^{22} &= E(-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_{k+1}} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k)) + \mathbf{C}_k, \end{aligned}$$

$$\mathbf{C}_k = \begin{cases} E(-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_{k+1}} \ln p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \\ \quad \mathbf{z}_k, \dots, \mathbf{z}_{k-l+1})), & l \geq 1 \\ E(-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_{k+1}} \ln p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})), & l = 0. \end{cases}$$

Proof: See the Appendix.

Remark 3.6. When the process noise and the measurement noise are mutually independent and temporally independent, respectively, i.e., $l_1 = 0, l_2 = 0, l_3 = 0, l_4 = 0$. Based on Corollary 3.5, it is easy to see that the recursion is the same as Proposition 1 in [17].

4 Numerical Examples

In this section, we consider two target tracking examples when noises of dynamic systems are temporally correlated. We compare the new PCRB to already existing techniques which include the method of [17], the pre-whitening method, the state augmentation method and the unbiased measurement conversion method given in [4]. Moreover, based on the PCRB, we can consider a sensor selection problem, i.e., determine how many sensors should be selected to obtain a desired tracking performance (see, e.g., [32, 33]).

4.1 Example 1

Consider a discrete time second order kinematic system driven by temporally correlated noises. This “correlated noise acceleration model” can be used in maneuvering tracking [1, 34]. The discrete time state equation is

$$\mathbf{x}_{k+1} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \mathbf{x}_k + \omega_k, \quad (41)$$

where the process noise is an one-step correlated moving-average model, i.e.,

$$\omega_k = \tilde{\omega}_k + 0.2\tilde{\omega}_{k-1} \quad (42)$$

$\{\tilde{\omega}_k\}$ is a Gaussian white noise with zero mean and variance $\tilde{\mathbf{Q}}_k = \begin{pmatrix} \frac{T^3}{3} & \frac{T^2}{2} \\ \frac{T^2}{2} & T \end{pmatrix} \mathbf{q}$, with power spectral density $q = 10$ and sampling interval $T = 2$.

The measurement is given by

$$\mathbf{z}_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}_k + \nu_k, \quad (43)$$

where measurement noise is considered one-step correlated and one-step cross-correlated with process noise as discussed in [2, 3], i.e.,

$$\nu_k = \tilde{\nu}_k + 0.2\tilde{\nu}_{k-1} + \omega_{k-1}, \quad (44)$$

where $\{\tilde{\nu}_k\}$ is a Gaussian white noise with zero mean and variance $\tilde{\mathbf{R}}_k = \begin{pmatrix} 20^2 & 0 \\ 0 & 5^2 \end{pmatrix}$; $\{\tilde{\nu}_k\}$ and $\{\tilde{\omega}_k\}$ are mutually independent.

By (41)-(44), it can easily be shown that $\{\nu_k\}$ is one-step correlated, $\{\omega_k\}$ is one-step correlated, $\{\nu_k\}$ is backward two-step and forward one-step correlated with $\{\omega_k\}$, i.e., $l_1 = 1, l_2 = 1, l_3 = 2, l_4 = 1$. Thus, Theorem 3.1 can be evaluated.

From these assumptions, the conditional probability densities are given as

$$\begin{aligned}
& -\ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \mathbf{z}_k) \\
& = c_1 + \frac{1}{2}[\mathbf{x}_{k+1} - g_k(\mathbf{x}_k, \mathbf{z}_k)]^T \tilde{\mathbf{Q}}_k^{-1}[\mathbf{x}_{k+1} - g_k(\mathbf{x}_k, \mathbf{z}_k)], \\
& -\ln p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \mathbf{z}_k, \mathbf{x}_k) \\
& = c_2 + \frac{1}{2}[\mathbf{z}_{k+1} - e_k(\mathbf{x}_{k+1}, \mathbf{z}_k, \mathbf{x}_k)]^T \\
& \quad \cdot \tilde{\mathbf{R}}_{k+1}^{-1}[\mathbf{z}_{k+1} - e_k(\mathbf{x}_{k+1}, \mathbf{z}_k, \mathbf{x}_k)],
\end{aligned}$$

where c_1 and c_2 are constants, and

$$\begin{aligned}
g_k(\mathbf{x}_k, \mathbf{z}_k) &= \begin{pmatrix} 0.8 & T \\ 0 & 0.8 \end{pmatrix} \mathbf{x}_k + 0.2\mathbf{z}_k \\
&\quad - 0.2\tilde{\nu}_k - 0.2^2\tilde{\nu}_{k-1} - 0.2^2\tilde{\omega}_{k-2}, \\
e_k(\mathbf{x}_{k+1}, \mathbf{z}_k, \mathbf{x}_k) &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}_{k+1} - \begin{pmatrix} 1.2 & T \\ 0 & 1.2 \end{pmatrix} \mathbf{x}_k \\
&\quad + 0.2\mathbf{z}_k - 0.2^2\tilde{\nu}_{k-1} - 0.2\omega_{k-1}.
\end{aligned} \tag{45}$$

Using (16)-(17), we can get

$$\begin{aligned}
\mathbf{B}_k^{11} &= \begin{pmatrix} 0.8 & T \\ 0 & 0.8 \end{pmatrix}^T \tilde{\mathbf{Q}}_k^{-1} \begin{pmatrix} 0.8 & T \\ 0 & 0.8 \end{pmatrix}, \\
\mathbf{C}_k^{11} &= \begin{pmatrix} 1.2 & T \\ 0 & 1.2 \end{pmatrix}^T \tilde{\mathbf{R}}_{k+1}^{-1} \begin{pmatrix} 1.2 & T \\ 0 & 1.2 \end{pmatrix}, \\
\mathbf{B}_k^{12} &= -\begin{pmatrix} 0.8 & T \\ 0 & 0.8 \end{pmatrix}^T \tilde{\mathbf{Q}}_k^{-1}, \\
\mathbf{C}_k^{12} &= -\begin{pmatrix} 1.2 & T \\ 0 & 1.2 \end{pmatrix}^T \tilde{\mathbf{R}}_{k+1}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \\
\mathbf{B}_k^{22} &= \tilde{\mathbf{Q}}_k^{-1}, \quad \mathbf{C}_k^{22} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^T \tilde{\mathbf{R}}_{k+1}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.
\end{aligned}$$

A straightforward calculation of (26)-(28) gives

$$\begin{aligned}
\mathbf{D}_k^{11} &= \mathbf{B}_k^{11} + \mathbf{C}_k^{11}, \\
\mathbf{D}_k^{12} &= \mathbf{B}_k^{12} + \mathbf{C}_k^{12}, \\
\mathbf{D}_k^{22} &= \mathbf{B}_k^{22} + \mathbf{C}_k^{22}.
\end{aligned}$$

Thus, we can derive the PCRB for estimating state vector $\{\mathbf{x}_k\}$ by (14) and (25)-(28) of Theorem 3.1. The corresponding PCRB is denoted by PCRB-T in Figures 1–3.

Since there are no existing techniques to handle auto-correlation and cross-correlation simultaneously, we approximatively use the state augmentation method, the pre-whitening method, and the method of [17].

For the state augmentation method, we consider auto-correlation of measurement noise and state noise simultaneously but ignoring the cross-correlation. By (42), we can easily get $\omega_k = 0.2\omega_{k-1} + \tilde{\omega}_k - 0.2^2\tilde{\omega}_{k-2}$, which is not an auto-regressive model. If we let $\beta_k = \tilde{\omega}_k - 0.2^2\tilde{\omega}_{k-2}$ which is a Gaussian white noise with zero mean and variance $\tilde{\mathbf{Q}}_k + 0.2^4\tilde{\mathbf{Q}}_{k-2}$, and assume that $\{\beta_k\}$ are mutually independent, then the state noise can be approximated by an auto-regressive model $\omega_k = 0.2\omega_{k-1} + \beta_k$. Similarly, by (44), $\nu_k = 0.2\nu_{k-1} + \tilde{\nu}_k - 0.2^2\tilde{\nu}_{k-2} + \tilde{\omega}_{k-1} - 0.2^2\tilde{\omega}_{k-3}$, let $\gamma_k = \tilde{\nu}_k - 0.2^2\tilde{\nu}_{k-2} + \tilde{\omega}_{k-1} - 0.2^2\tilde{\omega}_{k-3}$ which is a Gaussian white noise with zero mean and variance $\tilde{\mathbf{Q}}_{k-1} + 0.2^4\tilde{\mathbf{Q}}_{k-3} + \tilde{\mathbf{R}}_k + 0.2^4\tilde{\mathbf{R}}_{k-2}$, and assume that $\{\gamma_k\}$ are mutually independent, then the measurement noise can be approximated by an auto-regressive model $\nu_k = 0.2\nu_{k-1} + \gamma_k$. Therefore, based on the auto-regressive models $\omega_k = 0.2\omega_{k-1} + \beta_k$ and $\nu_k = 0.2\nu_{k-1} + \gamma_k$, we can use the the state augmentation method given in [4] to derive an approximate PCRB which is denoted by PCRB-A in Figures 1–3.

For the pre-whitening method, we consider cross-correlation of measurement noise and state noise but ignoring auto-correlation of them. Therefore, we can use the pre-whitening method given in [4] to derive an approximate PCRB which is denoted by PCRB-P in Figures 1–3.

For the method of [17], we ignore the correlation of noises and assume independent noises. Thus, we can use the method of [17] to derive an approximate PCRB which is denoted by PCRB-I in Figures 1–3.

PCRBs of the position and velocity state are plotted as a function of the time step in Figures 1–2, respectively. For sensor selection, the average PCRB of 40 time steps is plotted as a function of number of selected sensors in Figure 3.

The Figures 1-2 show that the new PCRB is significantly different from those of the other methods. The reason maybe that the approximation loss of the augmentation method and the pre-whitening method which ignore parts of correlation of noises and cannot deal with auto-correlation and cross-correlation simultaneously. In addition, Figures 1-2 show that the time-invariant character of the kinematic model implies that the PCRB converges to a constant after some time steps. In Figure 3, it can be seen that when the number of selected sensors is increasing, the gap of PCRB becomes smaller. Figure 3 also shows that if we want to achieve PCRB of the estimation error less than 30 m , 6 sensors have to be used based on the new PCRB at least. However, if we only consider the case of the auto-correlation of the state and measurement noises, 7 sensors have to be used, the other cases may be used more than 8 sensors. Thus, number of selected sensors becomes very different to obtain a desired estimation performance.

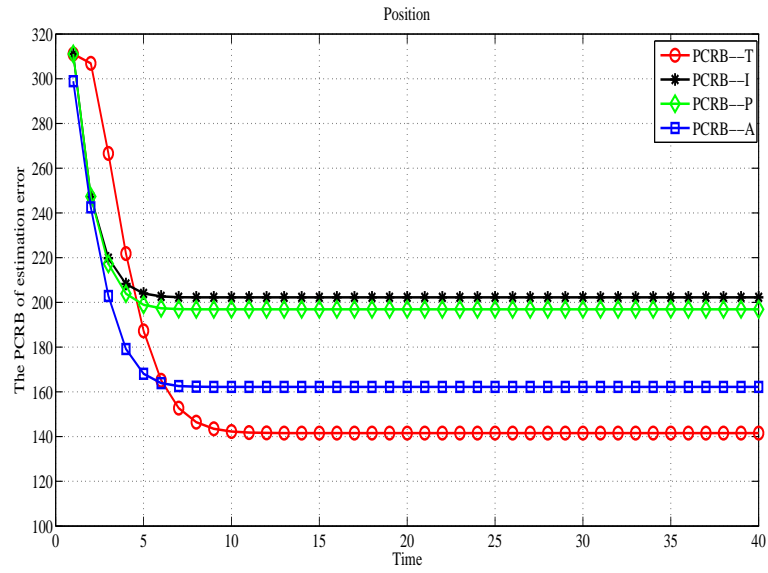


Figure 1: The PCRB of the position is plotted as a function of the time step.

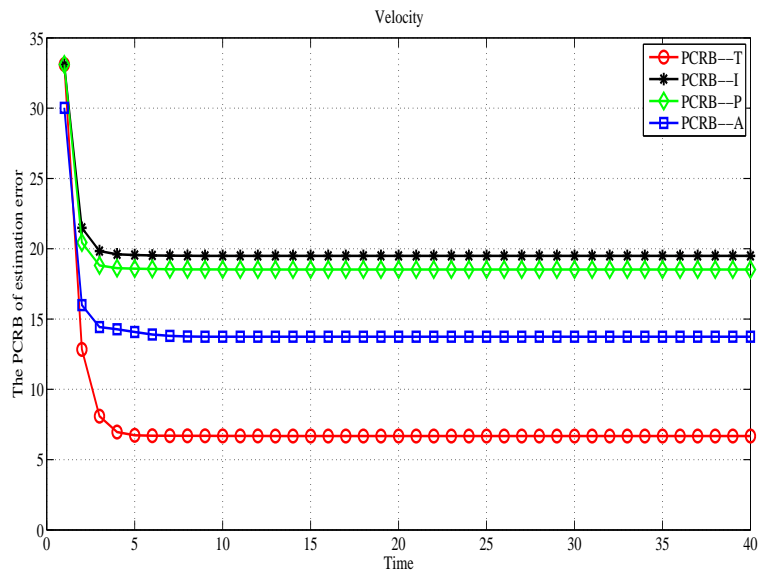


Figure 2: The PCRB of the velocity is plotted as a function of the time step.

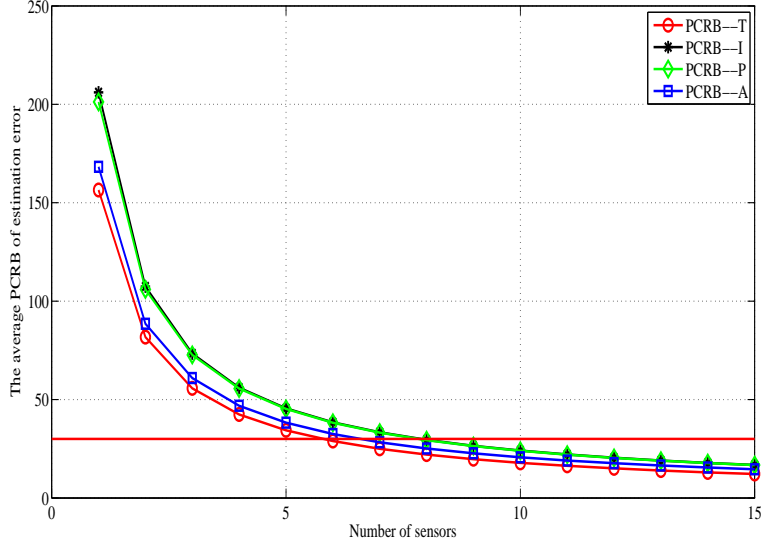


Figure 3: The average PCRB of 40 time step of the position is plotted as a function of number of selected sensors.

4.2 Example 2

In this example, we consider a discrete time dynamic system with *nonlinear* measurements as follows. The four-dimensional state variable includes position and velocity (x, \dot{x}, y, \dot{y}) driven by correlated noise, respectively,

$$\mathbf{x}_{k+1} = \mathbf{F}_k \mathbf{x}_k + \omega_k, \quad \mathbf{F}_k = \begin{pmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (46)$$

where the process noise is an two-step correlated moving-average model,

$$\omega_k = \tilde{\omega}_k + \tilde{\omega}_{k-1} + \tilde{\omega}_{k-2}. \quad (47)$$

$\{\tilde{\omega}_k\}$ is a Gaussian white noise with zero mean and variance

$$\tilde{\Sigma}_1 = \begin{pmatrix} \frac{T^3}{3} & \frac{T^2}{2} & 0 & 0 \\ \frac{T^2}{2} & T & 0 & 0 \\ 0 & 0 & \frac{T^3}{3} & \frac{T^2}{2} \\ 0 & 0 & \frac{T^2}{2} & T \end{pmatrix} q, \quad (48)$$

with sampling interval $T = 3$ and power spectral density $q = 10$.

The two-dimensional nonlinear measurement vector includes range and azimuth, respectively,

$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k, \quad (49)$$

where the nonlinear measurement function is

$$\mathbf{h}_k(\mathbf{x}_k) = \begin{pmatrix} \mathbf{h}_k^1(\mathbf{x}_k) \\ \mathbf{h}_k^2(\mathbf{x}_k) \end{pmatrix} = \begin{pmatrix} \sqrt{(\mathbf{x}_k^1)^2 + (\mathbf{x}_k^3)^2} \\ \tan^{-1}(\frac{\mathbf{x}_k^3}{\mathbf{x}_k^1}) \end{pmatrix}. \quad (50)$$

\mathbf{x}_k^i is the i -th entry of the state vector \mathbf{x}_k . $\{\mathbf{v}_k\}$ is a Gaussian white noise with zero mean and variance matrix

$$\Sigma_2 = \begin{pmatrix} 50^2 & 0 \\ 0 & 0.01^2 \end{pmatrix}. \quad (51)$$

$\{\mathbf{v}_k\}$ and $\{\tilde{\omega}_k\}$ are mutually independent.

It can easily be seen that the model satisfies Corollary 3.4 and $l = 2$. From these assumptions, the conditional probability densities are given as

$$\begin{aligned} & -\ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \mathbf{x}_{k-1}) \\ & = c_3 + (\mathbf{x}_{k+1} - g(\mathbf{x}_k, \mathbf{x}_{k-1}))^T \tilde{\Sigma}_1^{-1} (\mathbf{x}_{k+1} - g(\mathbf{x}_k, \mathbf{x}_{k-1})), \\ & -\ln p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}) \\ & = c_4 + (\mathbf{z}_{k+1} - h(\mathbf{x}_{k+1}))^T \Sigma_2^{-1} (\mathbf{z}_{k+1} - h(\mathbf{x}_{k+1})), \\ & g(\mathbf{x}_k, \mathbf{x}_{k-1}) \\ & = (\mathbf{I} + \mathbf{F}_k)\mathbf{x}_k - \mathbf{F}_k\mathbf{x}_{k-1} - \tilde{\omega}_{k-3}, \end{aligned}$$

where c_3 and c_4 are constants; \mathbf{I} is an identity matrix with compatible dimensions. A straightforward calculation of (35)-(36) gives

$$\begin{aligned} \mathbf{B}_k^{11} &= \mathbf{F}_k^T \tilde{\Sigma}_1^{-1} \mathbf{F}_k, \quad \mathbf{B}_k^{12} = -\mathbf{F}_k^T \tilde{\Sigma}_1^{-1} (\mathbf{I} + \mathbf{F}_k), \\ \mathbf{B}_k^{13} &= \mathbf{F}_k^T \tilde{\Sigma}_1^{-1}, \quad \mathbf{B}_k^{21} = (\mathbf{B}_k^{12})^T, \\ \mathbf{B}_k^{22} &= (\mathbf{I} + \mathbf{F}_k)^T \tilde{\Sigma}_1^{-1} (\mathbf{I} + \mathbf{F}_k), \quad \mathbf{B}_k^{23} = -(\mathbf{I} + \mathbf{F}_k)^T \tilde{\Sigma}_1^{-1}, \\ \mathbf{B}_k^{31} &= (\mathbf{B}_k^{13})^T, \quad \mathbf{B}_k^{32} = (\mathbf{B}_k^{23})^T, \quad \mathbf{B}_k^{33} = \tilde{\Sigma}_1^{-1}, \\ \mathbf{C}_k^{11} &= E\{[\nabla_{\mathbf{x}_{k+1}} h(\mathbf{x}_{k+1})]^T \Sigma_2^{-1} [\nabla_{\mathbf{x}_{k+1}} h(\mathbf{x}_{k+1})]^T\}. \end{aligned}$$

\mathbf{C}_k^{11} can be calculated by numerical Monte-Carlo methods. Using (37)-(39), we can easily get

$$\begin{aligned} \mathbf{D}_k^{11} &= \begin{pmatrix} \mathbf{B}_k^{11} & \mathbf{B}_k^{12} \\ \mathbf{B}_k^{21} & \mathbf{B}_k^{22} \end{pmatrix}, \\ \mathbf{D}_k^{21} &= \begin{pmatrix} \mathbf{B}_k^{31} & \mathbf{B}_k^{32} \end{pmatrix} = (\mathbf{D}_k^{12})^T, \\ \mathbf{D}_k^{22} &= \mathbf{B}_k^{33} + \mathbf{C}_k^{11}. \end{aligned} \quad (52)$$

Combing (52), (33) and (34), after some simplification, we have the simpler recursion

$$\begin{aligned}
\mathbf{J}_{k+1} &= \mathbf{D}_k^{22} - \mathbf{D}_k^{21}(\mathbf{E}_k + \mathbf{D}_k^{11})^{-1}\mathbf{D}_k^{12}, \\
\mathbf{E}_k^{11} &= \mathbf{E}_{k-1}^{22} + \mathbf{B}_{k-1}^{22} - (\mathbf{E}_{k-1}^{21} + \mathbf{B}_{k-1}^{21}) \\
&\quad \cdot (\mathbf{E}_{k-1}^{11} + \mathbf{B}_{k-1}^{11})^{-1}(\mathbf{E}_{k-1}^{12} + \mathbf{B}_{k-1}^{12}), \\
\mathbf{E}_k^{12} &= \mathbf{B}_{k-1}^{23} - (\mathbf{E}_{k-1}^{21} + \mathbf{B}_{k-1}^{21})(\mathbf{E}_{k-1}^{11} + \mathbf{B}_{k-1}^{11})^{-1}\mathbf{B}_{k-1}^{13} \\
&= (\mathbf{E}_k^{21})^T, \\
\mathbf{E}_k^{22} &= \mathbf{B}_{k-1}^{33} + \mathbf{C}_{k-1}^{11} - \mathbf{B}_{k-1}^{31}(\mathbf{E}_{k-1}^{11} + \mathbf{B}_{k-1}^{11})^{-1}\mathbf{B}_{k-1}^{13}.
\end{aligned}$$

Thus, based on the above recursion, we can derive the PCRB for estimating state vector $\{\mathbf{x}_k\}$ by Corollary 3.4. The corresponding PCRB is denoted by PCRB-C in Figures 4–5.

Since there are no existing methods to deal with this example accurately, we approximatively use the unbiased measurement conversion method given in [4], which can convert the nonlinear system into linear system. Moreover, similar to Example 1, we use the state augmentation method to derive an approximate PCRB which is denoted by PCRB-CA in Figures 4–5. In addition, for the converted linear system, we can also use Corollary 3.4 to derive an approximate PCRB which is denoted by PCRB-CC in Figures 4–5.

In Figure 4, PCRB of the position is plotted as a function of the time step. For sensor selection, the *average* PCRB of 40 time steps is plotted as a function of number of selected sensors in Figure 5.

Figures 4-5 shows that the new PCRB is significantly different from the other two methods. The reason maybe the approximation loss of unbiased conversion, and that the pdf of the noise of the converted linear system is non-Gaussian and uncertain, and the approximation loss of the augmentation method which ignore parts of correlation of noises. Figure 5 also shows that if we want to achieve PCRB of the estimation error less than 100 m , 6 sensors have to used at least. However, if we use the other methods, we have to select 11 sensors. Thus, this example also shows that number of selected sensors becomes very different to obtain a desired estimation performance.

5 Conclusions

In this paper, we have derived a unified recursive formula of the mean-square error lower bound for the discrete-time nonlinear filtering problem when noises of dynamic systems are temporally correlated based on the posterior version of the Cramér-Rao inequality. It can be applicable to the multi-step correlated process noise, multi-step correlated measurement noise and multi-step cross-correlated process and measurement noise simultaneously. Although the unified formula can come across the three cases of finite-step correlated noises, a few corollaries with simpler formulae follow to elucidate special cases, which may be used more frequently. Two typical target tracking examples have shown that the new PCRB is significantly different from that of the other existing approximation methods which include the method of ignoring correlation of

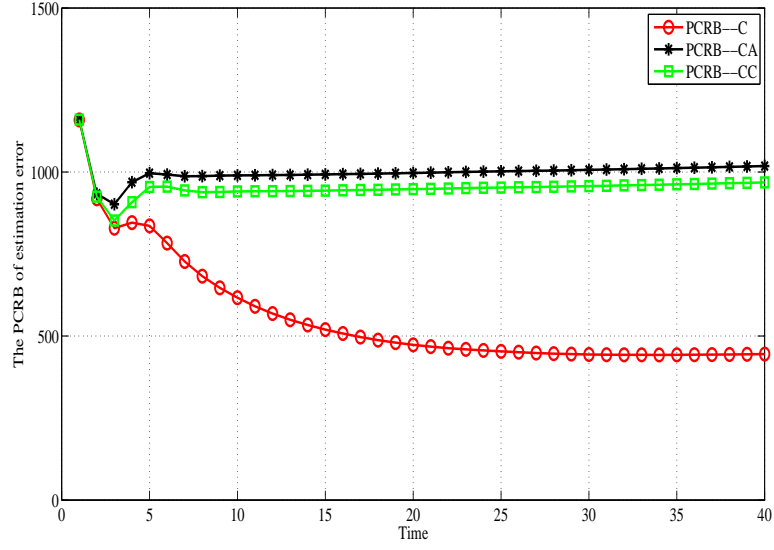


Figure 4: The PCRB of the position is plotted as a function of the time step.

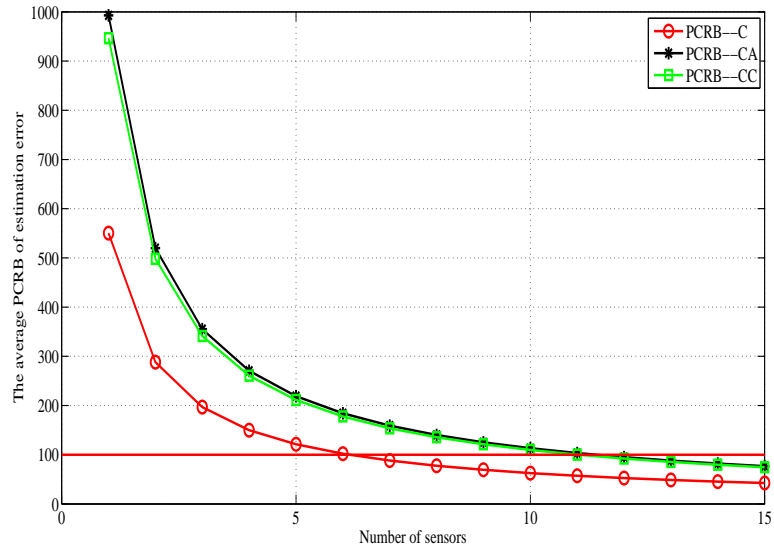


Figure 5: The average PCRB of 40 time steps is plotted as a function of number of selected sensors.

noises, the pre-whitening method, the state augmentation method and the unbiased measurement conversion method. Thus, when they are applied to sensor selection problems, simulations show that number of selected sensors becomes very different to obtain a desired estimation performance. Future research challenges include sensor management when noises of dynamic systems are temporally correlated for multitarget tracking and data association.

6 Appendix

Lemma 6.1. (see, e.g., [35, 13]) Consider a partitioned matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}.$$

If \mathbf{A} is invertible, then \mathbf{A}_{11} is invertible if and only if the Schur complement $\Delta = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$ is invertible, and

$$\begin{aligned} \mathbf{A}^{-1} &= \begin{pmatrix} I & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathbf{A}_{11}^{-1} & 0 \\ 0 & \Delta^{-1} \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} I & 0 \\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & I \end{pmatrix}. \end{aligned}$$

Lemma 6.2. (see, e.g., [35, 13]) Let $\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{pmatrix}$. If \mathbf{A} and \mathbf{A}_{11} are both invertible, then $\Delta = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$ is invertible, and use the Lemma 6.1, we have

$$\begin{aligned} \mathbf{B}\mathbf{A}^{-1}\mathbf{C} &= \mathbf{B}_{11}\mathbf{A}_{11}^{-1}\mathbf{C}_{11} + (\mathbf{B}_{12} - \mathbf{B}_{11}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})\Delta^{-1} \\ &\quad \cdot (\mathbf{C}_{21} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{C}_{11}). \end{aligned} \tag{53}$$

6.1 proof of Proposition 3.1

According to the three definitions of *finite-step correlated* noises given in Section 2.1 and (1)-(2), to derive a recursive formula, $p(\mathbf{x}_{k+1}|\mathbf{X}_k, \mathbf{Z}_k)$ and $p(\mathbf{z}_{k+1}|\mathbf{X}_{k+1}, \mathbf{Z}_k)$ of (8) may be approximately written as

$$p(\mathbf{x}_{k+1}|\mathbf{X}_k, \mathbf{Z}_k) \approx$$

$$p(\mathbf{x}_{k+1}|\mathbf{x}_k) \begin{cases} p(\mathbf{x}_{k+1}|\mathbf{x}_k) & \text{for } l_2 = 0 \text{ or } l_2 = 1 \\ & \text{and } l_4 = 0 \\ p(\mathbf{x}_{k+1}|\mathbf{x}_k, \dots, \mathbf{x}_{k-l_2+1}) & \text{for } l_2 \geq 2, l_4 = 0 \\ p(\mathbf{x}_{k+1}|\mathbf{x}_k, \\ \quad \mathbf{z}_k, \dots, \mathbf{z}_{k-l_4+1}) & \text{for } l_2 = 0 \\ & \text{or } l_2 = 1 \text{ and } l_4 > 0 \\ p(\mathbf{x}_{k+1}|\mathbf{x}_k, \dots, \mathbf{x}_{k-l_2+1}, \\ \quad \mathbf{z}_k, \dots, \mathbf{z}_{k-l_4+1}) & \text{for } l_2 \geq 2, l_4 > 0, \end{cases} \quad (54)$$

$$p(\mathbf{z}_{k+1}|\mathbf{X}_{k+1}, \mathbf{Z}_k) \approx$$

$$\begin{cases} p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}) & \text{for } l_3 = 0 \text{ or } l_3 = 1 \\ & \text{and } l_1 = 0 \\ p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \dots, \\ \quad \mathbf{x}_{k-l_3+2}) & \text{for } l_3 \geq 2, \\ & l_1 = 0 \\ p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \\ \quad \mathbf{z}_k, \dots, \mathbf{z}_{k-l_1+1}) & \text{for } l_3 = 0 \\ & \text{or } l_3 = 1 \text{ and } l_1 > 0 \\ p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \dots, \mathbf{x}_{k-l_3+2} \\ \quad \mathbf{z}_k, \dots, \mathbf{z}_{k-l_1+1}) & \text{for } l_3 \geq 2, l_1 > 0. \end{cases} \quad (55)$$

If we denote by $l'_3 \triangleq \max\{l_3, 1\}$, $l'_2 \triangleq \max\{l_2, 1\}$, then (54) and (55) can be simplified as

$$p(\mathbf{x}_{k+1}|\mathbf{X}_k, \mathbf{Z}_k) \approx \begin{cases} p(\mathbf{x}_{k+1}|\mathbf{x}_k, \dots, \mathbf{x}_{k-l'_2+1}) & \text{for } l_4 = 0 \\ p(\mathbf{x}_{k+1}|\mathbf{x}_k, \dots, \mathbf{x}_{k-l'_2+1}, \\ \quad \mathbf{z}_k, \dots, \mathbf{z}_{k-l_4+1}) & \text{for } l_4 > 0, \end{cases} \quad (56)$$

$$p(\mathbf{z}_{k+1}|\mathbf{X}_{k+1}, \mathbf{Z}_k) \approx \begin{cases} p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \dots, \mathbf{x}_{k-l'_3+2}) & \text{for } l_1 = 0 \\ p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \dots, \mathbf{x}_{k-l'_3+2}, \\ \quad \mathbf{z}_k, \dots, \mathbf{z}_{k-l_1+1}) & \text{for } l_1 > 0. \end{cases} \quad (57)$$

In order to derive the recursion of \mathbf{J}_{k+1} , we need to decompose vector \mathbf{X}_k and \mathbf{X}_{k+1} . Based on (56)-(57), we find that the decomposition depends on l'_3 and $l'_2 + 1$. Thus, we discuss three cases $l'_3 > l'_2 + 1$, $l'_3 < l'_2 + 1$ and $l'_3 = l'_2 + 1$, respectively.

1. $l'_3 > l'_2 + 1$

We decompose \mathbf{X}_k as

$$\mathbf{X}_k = (\mathbf{x}'_{k-l'_3+1}, \dots, \mathbf{x}'_{k-l'_2+1}, \dots, \mathbf{x}'_{k-1}, \mathbf{x}'_k)' \quad (58)$$

and $J(\mathbf{X}_k)$ correspondingly as

$$J(\mathbf{X}_k) =$$

$$\begin{pmatrix} \mathbf{A}_k^{1,1} & \dots & \mathbf{A}_k^{1,l'_3-l'_2} & \dots & \mathbf{A}_k^{1,l'_3} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \mathbf{A}_k^{l'_3-l'_2,1} & \dots & \mathbf{A}_k^{l'_3-l'_2,l'_3-l'_2} & \dots & \mathbf{A}_k^{l'_3-l'_2,l'_3} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \mathbf{A}_k^{l'_3,1} & \dots & \mathbf{A}_k^{l'_3,l'_3-l'_2} & \dots & \mathbf{A}_k^{l'_3,l'_3} \end{pmatrix}$$

where

$$\begin{aligned} \mathbf{A}_k^{1,1} &= E(-\Delta_{\mathbf{x}_{k-l'_3+1}}^{\mathbf{x}_{k-l'_3+1}} \ln p_k), \\ \mathbf{A}_k^{1,j} &= E(-\Delta_{\mathbf{x}_{k-l'_3+1}}^{\mathbf{x}_{k-l'_3+1}+j} \ln p_k), \\ \mathbf{A}_k^{i,1} &= E(-\Delta_{\mathbf{x}_{k-l'_3+i}}^{\mathbf{x}_{k-l'_3+i}} \ln p_k), \\ \mathbf{A}_k^{i,j} &= E(-\Delta_{\mathbf{x}_{k-l'_3+i}}^{\mathbf{x}_{k-l'_3+i}+j} \ln p_k), \\ &\text{for } i, j = 2, \dots, l'_3. \end{aligned} \tag{59}$$

Using (8), (56), (57) the posterior information matrix for \mathbf{X}_{k+1} can be written in block form as

$$J(\mathbf{X}_{k+1}) =$$

$$\begin{pmatrix} \mathbf{A}_k^{1,1} & \dots & \mathbf{A}_k^{1,l'_3} & 0 \\ \vdots & \dots & \dots & \vdots \\ \mathbf{A}_k^{l'_3,1} & \dots & \mathbf{A}_k^{l'_3,l'_3} & \mathbf{C}_k^{l'_{31},l'_3} \\ & & +\mathbf{C}_k^{l'_{31},l'_{31}} & +\mathbf{B}_k^{l'_2,l'_{21}} \\ & & +\mathbf{B}_k^{l'_2,l'_2} & \\ 0 & \dots & \mathbf{C}_k^{l'_{31},l'_{31}} & \mathbf{C}_k^{l'_3,l'_3} \\ & & +\mathbf{B}_k^{l'_{21},l'_2} & +\mathbf{B}_k^{l'_{21},l'_{21}} \end{pmatrix}$$

where 0's stand for the zero blocks of appropriate dimensions; $\mathbf{B}_k^{i,j}$ and $\mathbf{C}_k^{i,j}$ are defined in (16)-(17). To simplify the matrix $J(\mathbf{X}_{k+1})$, we denote by $l'_{32} \triangleq l'_3 - l'_2$, $l'_{321} \triangleq l'_3 - l'_2 + 1$, $l'_{31} \triangleq l'_3 - 1$, $l'_{21} \triangleq l'_2 + 1$.

Moreover, the information submatrix \mathbf{J}_{k+1} for estimating $\mathbf{x}_{k+1} \in R^r$ is given as the inverse of the

$(r \times r)$ right-lower block of $[J(\mathbf{X}_{k+1})]^{-1}$, i.e.,

$$\begin{aligned}
\mathbf{J}_{k+1} &= \mathbf{C}_k^{l'_3, l'_3} + \mathbf{B}_k^{l'_{21}, l'_{21}} \\
&- \begin{pmatrix} 0 & \mathbf{C}_k^{l'_3, 1} & \dots & \mathbf{C}_k^{l'_3, l'_3-1} \\ & & & +\mathbf{B}_k^{l'_2+1, l'_2} \end{pmatrix} \\
&\cdot \begin{pmatrix} \mathbf{A}_k^{1,1} & \mathbf{A}_k^{1,2} & \dots & \mathbf{A}_k^{1, l'_3} \\ \mathbf{A}_k^{2,1} & \mathbf{A}_k^{2,2} & \dots & \mathbf{A}_k^{2, l'_3} \\ & +\mathbf{C}_k^{1,1} & \dots & +\mathbf{C}_k^{1, l'_{31}} \\ \dots & \dots & \dots & \dots \\ \mathbf{A}_k^{l'_3, 1} & \mathbf{A}_k^{l'_3, 2} & \dots & \mathbf{A}_k^{l'_3, l'_3} \\ & +\mathbf{C}_k^{l'_{31}, 1} & & +\mathbf{C}_k^{l'_{31}, l'_{31}} \\ & & & +\mathbf{B}_k^{l'_2, l'_2} \end{pmatrix}^{-1} \\
&\cdot \begin{pmatrix} 0 & \mathbf{C}_k^{l'_3, 1} & \dots & \mathbf{C}_k^{l'_3, l'_3-1} \\ & & & +\mathbf{B}_k^{l'_2+1, l'_2} \end{pmatrix}^T. \tag{60}
\end{aligned}$$

Using Lemma 6.2 and (18)-(20), it follows that

$$\mathbf{J}_{k+1} = \mathbf{D}_k^{22} - \mathbf{D}_k^{21}(\mathbf{E}_k + \mathbf{D}_k^{11})^{-1}\mathbf{D}_k^{12}, \tag{61}$$

$$\begin{aligned}
\mathbf{E}_k^{i,j} &= \mathbf{A}_k^{i+1, j+1} - \mathbf{A}_k^{i+1, 1}(\mathbf{A}_k^{1,1})^{-1}\mathbf{A}_k^{1, j+1} \\
&= (\mathbf{E}_k^{j,i})', \\
&\text{for } i, j = 1, \dots, l'_3 - 1.
\end{aligned} \tag{62}$$

Since \mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} in (61) can be calculated as (18)-(20), we only need to derive the recursion of $\mathbf{E}_k^{i,j}$.

Based on the recursion between p_k and p_{k-1} given in (8) which depends on (56) and (57), and using the definitions of $\mathbf{A}_k^{i,j}$, $\mathbf{B}_k^{i,j}$, $\mathbf{C}_k^{i,j}$ given in (59), (16), (17), respectively, it follows that

$$\begin{aligned}
&\begin{pmatrix} \mathbf{A}_k^{1,1} & \mathbf{A}_k^{1, j+1} \\ \mathbf{A}_k^{i+1, 1} & \mathbf{A}_k^{i+1, j+1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}_{k-1}^{1,1} & \mathbf{A}_{k-1}^{1,2} & \mathbf{A}_{k-1}^{1, j+2} \\ \mathbf{A}_{k-1}^{2,1} & \mathbf{A}_{k-1}^{2,2} & \mathbf{A}_{k-1}^{2, j+2} \\ & +\mathbf{C}_{k-1}^{11} & +\mathbf{C}_{k-1}^{1, j+1} \\ \mathbf{A}_{k-1}^{i+2, 1} & \mathbf{A}_{k-1}^{i+2, 2} & \mathbf{A}_{k-1}^{i+2, j+2} \\ & +\mathbf{C}_{k-1}^{i+1, 1} & +\mathbf{C}_{k-1}^{i+1, j+1} \\ & & +\mathbf{B}_{k-1}^{i+2- l'_3+ l'_2, j+2- l'_3+ l'_2} \end{pmatrix}
\end{aligned}$$

(63)

$$\text{for } i = 1, \dots, l'_3 - 1, j = 1, \dots, l'_3 - 1.$$

where if $i \leq l'_3 - l'_2 - 2$ or $j \leq l'_3 - l'_2 - 2$, then $\mathbf{B}_{k-1}^{i+l'_2-l'_3+2, j+l'_2-l'_3+2} = 0$; if $i > l'_3$ or $j > l'_3$, then $\mathbf{A}_{k-1}^{i,j} = 0$.

Note that the matrix \mathbf{E}_k^{ij} in (62) is the Schur complement of the block $\mathbf{A}_k^{1,1}$ of the left matrix of Equation (63) and the Schur complement of the corresponding block of the right matrix of Equation (63) is

$$\begin{aligned} & \mathbf{A}_{k-1}^{i+2, j+2} + \mathbf{C}_{k-1}^{i+1, j+1} + \mathbf{B}_{k-1}^{i+2-l'_3+l'_2, j+2-l'_3+l'_2} \\ & - \begin{pmatrix} \mathbf{A}_{k-1}^{i+2, 1} & \mathbf{A}_{k-1}^{i+2, 2} + \mathbf{C}_{k-1}^{i+1, 1} \end{pmatrix} \\ & \cdot \begin{pmatrix} \mathbf{A}_{k-1}^{1, 1} & \mathbf{A}_{k-1}^{1, 2} \\ \mathbf{A}_{k-1}^{2, 1} & \mathbf{A}_{k-1}^{2, 2} + \mathbf{C}_{k-1}^{11} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{k-1}^{1, j+2} \\ \mathbf{A}_{k-1}^{2, j+2} + \mathbf{C}_{k-1}^{1, j+1} \end{pmatrix}. \end{aligned} \quad (64)$$

Using Lemma 6.2, we can simplify (64). Moreover, by the definition of \mathbf{E}_{k-1}^{ij} in (62), we have the recursion of $\mathbf{E}_k^{i,j}$ as

$$\begin{aligned} \mathbf{E}_k^{ij} &= \mathbf{E}_{k-1}^{i+1, j+1} + \mathbf{B}_{k-1}^{i+2-l'_3+l'_2, j+2-l'_3+l'_2} + \mathbf{C}_{k-1}^{i+1, j+1} \\ & - (\mathbf{E}_{k-1}^{i+1, 1} + \mathbf{C}_{k-1}^{i+1, 1})(\mathbf{E}_{k-1}^{11} + \mathbf{C}_{k-1}^{11})^{-1} \\ & \cdot (\mathbf{E}_{k-1}^{1, j+1} + \mathbf{C}_{k-1}^{1, j+1}). \end{aligned} \quad (65)$$

2. $l'_3 < l'_2 + 1$

We decompose \mathbf{X}_k as

$$\mathbf{X}_k = (\mathbf{x}'_{k-l'_2}, \dots, \mathbf{x}'_{k-l'_3+2}, \dots, \mathbf{x}'_{k-1}, \mathbf{x}'_k)'$$

and $J(\mathbf{X}_k)$ correspondingly as

$$J(\mathbf{X}_k) =$$

$$\begin{pmatrix} \mathbf{A}_k^{11} & \dots & \mathbf{A}_k^{1, l'_{233}} & \dots & \mathbf{A}_k^{1, l'_{21}} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \mathbf{A}_k^{l'_{233}, 1} & \dots & \mathbf{A}_k^{l'_{233}, l'_{233}} & \dots & \mathbf{A}_k^{l'_{233}, l'_{21}} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \mathbf{A}_k^{l'_{21}, 1} & \dots & \mathbf{A}_k^{l'_{21}, l'_{233}} & \dots & \mathbf{A}_k^{l'_{21}, l'_{21}} \end{pmatrix}$$

where

$$\begin{aligned}
\mathbf{A}_k^{1,1} &= E(-\Delta_{\mathbf{x}_{k-l'_2}}^{\mathbf{x}_{k-l'_2}} \ln p_k), \\
\mathbf{A}_k^{1,j} &= E(-\Delta_{\mathbf{x}_{k-l'_2}}^{\mathbf{x}_{k-l'_2}+j-1} \ln p_k), \\
\mathbf{A}_k^{i,1} &= E(-\Delta_{\mathbf{x}_{k-l'_2+i-1}}^{\mathbf{x}_{k-l'_2}} \ln p_k), \\
\mathbf{A}_k^{i,j} &= E(-\Delta_{\mathbf{x}_{k-l'_2+i-1}}^{\mathbf{x}_{k-l'_2}+j-1} \ln p_k), \\
&\text{for } i, j = 2, \dots, l'_2 + 1.
\end{aligned} \tag{66}$$

Using (8), (56), (57), the posterior information matrix for \mathbf{X}_{k+1} can be written in block form as

$$J(\mathbf{X}_{k+1}) =$$

$$\begin{pmatrix}
\mathbf{A}_k^{1,1} & \dots & \mathbf{A}_k^{1,l'_{21}} & 0 \\
\dots & \dots & \dots & \dots \\
\mathbf{A}_k^{l'_{21},1} & \dots & \mathbf{A}_k^{l'_{21},l'_{21}} & \mathbf{C}_k^{l'_{31},l'_3} \\
& & + \mathbf{C}_k^{l'_{31},l'_3} & + \mathbf{B}_k^{l'_2,l'_{21}} \\
& & + \mathbf{B}_k^{l'_2,l'_2} & \\
0 & \dots & \mathbf{C}_k^{l'_3,l'_{31}} & \mathbf{C}_k^{l'_3,l'_3} \\
& & + \mathbf{B}_k^{l'_{21},l'_2} & + \mathbf{B}_k^{l'_{21},l'_{21}}
\end{pmatrix}$$

where $\mathbf{B}_k^{i,j}$ and $\mathbf{C}_k^{i,j}$ are defined in (16)-(17). To simplify the matrix $J(\mathbf{X}_{k+1})$, we denote by $l'_{233} \triangleq l'_2 - l'_3 + 3$, $l'_{232} \triangleq l'_2 - l'_3 + 2$, $l'_{31} \triangleq l'_3 - 1$, $l'_{21} \triangleq l'_2 + 1$. The information submatrix \mathbf{J}_{k+1} for estimating $\mathbf{x}_{k+1} \in R^r$ is given as the inverse of the $(r \times r)$ right-lower block of $[J(\mathbf{X}_{k+1})]^{-1}$, i.e.,

$$\mathbf{J}_{k+1} = \mathbf{D}_k^{22} - \mathbf{D}_k^{21}(\mathbf{E}_k + \mathbf{D}_k^{11})^{-1}\mathbf{D}_k^{12}, \tag{67}$$

$$\begin{aligned}
\mathbf{E}_k^{i,j} &= \mathbf{A}_k^{i+1,j+1} - \mathbf{A}_k^{i+1,1}(\mathbf{A}_k^{11})^{-1}\mathbf{A}_k^{1,j+1} \\
&= (\mathbf{E}_k^{ji})', \\
&\text{for } i, j = 1, \dots, l'_2,
\end{aligned} \tag{68}$$

Since \mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} in (67) can be calculated as (18)-(20), we only needs to derive the recursion $\mathbf{E}_k^{i,j}$.

Based on the recursion between p_k and p_{k-1} given in (8) which depends on (56) and (57), and using the definitions of $\mathbf{A}_k^{i,j}$, $\mathbf{B}_k^{i,j}$, $\mathbf{C}_k^{i,j}$ given in (66), (16), (17), respectively, it follows that

$$\begin{pmatrix}
\mathbf{A}_k^{11} & \mathbf{A}_k^{1,j+1} \\
\mathbf{A}_k^{i+1,1} & \mathbf{A}_k^{i+1,j+1}
\end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{A}_{k-1}^{11} & \mathbf{A}_{k-1}^{12} & \mathbf{A}_{k-1}^{1,j+2} \\ \mathbf{A}_{k-1}^{21} & \mathbf{A}_{k-1}^{22} & \mathbf{A}_{k-1}^{2,j+2} \\ & +\mathbf{B}_{k-1}^{11} & +\mathbf{B}_{k-1}^{1,j+1} \\ \mathbf{A}_{k-1}^{i+2,1} & \mathbf{A}_{k-1}^{i+2,2} & \mathbf{A}_{k-1}^{i+2,j+2} \\ & +\mathbf{B}_{k-1}^{i+1,1} & +\mathbf{B}_{k-1}^{i+1,j+1} \\ & & +\mathbf{C}_{k-1}^{i-l'_2+l'_3,j-l'_2+l'_3} \end{pmatrix}, \quad (69)$$

for $i = 1, \dots, l'_2, j = 1, \dots, l'_2$,

where if $i \leq l'_2 - l'_3$ or $j \leq l'_2 - l'_3$, then $\mathbf{C}_{k-1}^{i-l'_2+l'_3,j-l'_2+l'_3} = 0$; if $i > l'_2 + 1$ or $j > l'_2 + 1$, then $\mathbf{A}_{k-1}^{i,j} = 0$. Moreover, similar to the derivation of Equation (65), we have the recursion of the matrix \mathbf{E}_k as

$$\begin{aligned} \mathbf{E}_k^{ij} &= \mathbf{E}_{k-1}^{i+1,j+1} + \mathbf{C}_{k-1}^{i-l'_2+l'_3,j-l'_2+l'_3} + \mathbf{B}_{k-1}^{i+1,j+1} \\ &\quad - (\mathbf{E}_{k-1}^{i+1,1} + \mathbf{C}_{k-1}^{i+1,1})(\mathbf{E}_{k-1}^{11} + \mathbf{C}_{k-1}^{11})^{-1} \\ &\quad \cdot (\mathbf{E}_{k-1}^{1,j+1} + \mathbf{C}_{k-1}^{1,j+1}). \end{aligned} \quad (70)$$

3. $l'_2 + 1 = l'_3$

We decompose \mathbf{X}_k as $\mathbf{X}_k = (\mathbf{X}'_{k-l'_3+1}, \dots, \mathbf{x}'_{k-1}, \mathbf{x}'_k)'$ and $J(\mathbf{X}_k)$ correspondingly as

$$J(\mathbf{X}_k) = \begin{pmatrix} \mathbf{A}_k^{11} & \dots & \mathbf{A}_k^{1,l'_3} \\ \vdots & \dots & \vdots \\ \mathbf{A}_k^{l'_3,1} & \dots & \mathbf{A}_k^{l'_3,l'_3} \end{pmatrix}$$

where

$$\begin{aligned} \mathbf{A}_k^{1,1} &= E(-\Delta_{\mathbf{x}_{k-l'_3+1}}^{\mathbf{x}_{k-l'_3+1}} \ln p_k), \\ \mathbf{A}_k^{1,j} &= E(-\Delta_{\mathbf{x}_{k-l'_3+1}}^{\mathbf{x}_{k-l'_3+1}} \ln p_k), \\ \mathbf{A}_k^{i,1} &= E(-\Delta_{\mathbf{x}_{k-l'_3+i}}^{\mathbf{x}_{k-l'_3+1}} \ln p_k), \\ \mathbf{A}_k^{i,j} &= E(-\Delta_{\mathbf{x}_{k-l'_3+i}}^{\mathbf{x}_{k-l'_3+1}} \ln p_k), \\ &\quad \text{for } i, j = 2, \dots, l'_3. \end{aligned} \quad (71)$$

Using (8), (56), (57), the posterior information matrix for \mathbf{X}_{k+1} can be written in block form as

$$J(\mathbf{X}_{k+1}) =$$

$$\begin{pmatrix} \mathbf{A}_k^{1,1} & \mathbf{A}_k^{1,2} & \dots & \mathbf{A}_k^{1,l'_3} & 0 \\ \mathbf{A}_k^{2,1} & \mathbf{A}_k^{2,2} & \dots & \mathbf{A}_k^{2,l'_3} & \mathbf{C}_k^{1,l'_3} \\ & +\mathbf{C}_k^{1,1} & & +\mathbf{C}_k^{1,l'_3-1} & +\mathbf{B}_k^{1,l'_3} \\ & +\mathbf{B}_k^{1,1} & & +\mathbf{B}_k^{1,l'_3-1} & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{A}_k^{l'_3,1} & \mathbf{A}_k^{l'_3,2} & \dots & \mathbf{A}_k^{l'_3,l'_3} & \mathbf{C}_k^{l'_3-1,l'_3} \\ & +\mathbf{C}_k^{l'_3-1,1} & & +\mathbf{C}_k^{l'_3-1,l'_3-1} & +\mathbf{B}_k^{l'_3-1,l'_3} \\ & +\mathbf{B}_k^{l'_3-1,1} & & +\mathbf{B}_k^{l'_3-1,l'_3-1} & \\ 0 & \mathbf{C}_k^{l'_3,1} & \dots & \mathbf{C}_k^{l'_3,l'_3-1} & \mathbf{C}_k^{l'_3,l'_3} \\ & +\mathbf{B}_k^{l'_3,1} & & +\mathbf{B}_k^{l'_3,l'_3-1} & +\mathbf{B}_k^{l'_3,l'_3} \end{pmatrix}$$

where $\mathbf{B}_k^{i,j}$ and $\mathbf{C}_k^{i,j}$ are defined in (16)-(17). The information submatrix \mathbf{J}_{k+1} for estimating $\mathbf{x}_{k+1} \in R^r$ is given as the inverse of the $(r \times r)$ right-lower block of $[J(\mathbf{X}_{k+1})]^{-1}$, i.e.,

$$\mathbf{J}_{k+1} = \mathbf{D}_k^{22} - \mathbf{D}_k^{21}(\mathbf{E}_k + \mathbf{D}_k^{11})^{-1}\mathbf{D}_k^{12}, \quad (72)$$

$$\begin{aligned} \mathbf{E}_k^{ij} &= \mathbf{A}_k^{i+1,j+1} - \mathbf{A}_k^{i+1,1}(\mathbf{A}_k^{11})^{-1}\mathbf{A}_k^{1,j+1}, \\ &= (\mathbf{E}_k^{ji})', \\ &\text{for } i, j = 1, \dots, l'_3 - 1. \end{aligned} \quad (73)$$

Since \mathbf{D}_k^{11} , \mathbf{D}_k^{12} , \mathbf{D}_k^{21} and \mathbf{D}_k^{22} in (72) can be calculated as (26)-(28), we only needs to derive the recursion $\mathbf{E}_k^{i,j}$.

Based on the recursion between p_k and p_{k-1} given in (8) which depends on (56) and (57), and using the definitions of $\mathbf{A}_k^{i,j}$, $\mathbf{B}_k^{i,j}$, $\mathbf{C}_k^{i,j}$ given in (71), (16), (17), respectively, it follows that

$$\begin{pmatrix} \mathbf{A}_k^{11} & \mathbf{A}_k^{1,j+1} \\ \mathbf{A}_k^{i+1,1} & \mathbf{A}_k^{i+1,j+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{k-1}^{11} & \mathbf{A}_{k-1}^{12} & \mathbf{A}_{k-1}^{1,j+2} \\ \mathbf{A}_{k-1}^{21} & \mathbf{A}_{k-1}^{22} & \mathbf{A}_{k-1}^{2,j+2} \\ & +\mathbf{C}_{k-1}^{11} & +\mathbf{C}_{k-1}^{1,j+1} \\ & +\mathbf{B}_{k-1}^{11} & +\mathbf{B}_{k-1}^{1,j+1} \\ \mathbf{A}_{k-1}^{i+2,1} & \mathbf{A}_{k-1}^{i+2,2} & \mathbf{A}_{k-1}^{i+2,j+2} \\ & +\mathbf{C}_{k-1}^{i+1,1} & +\mathbf{C}_{k-1}^{i+1,j+1} \\ & +\mathbf{B}_{k-1}^{i+1,1} & +\mathbf{B}_{k-1}^{i+1,j+1} \end{pmatrix},$$

for $i = 1, \dots, l'_3 - 1, j = 1, \dots, l'_3 - 1$.

where $\mathbf{A}_{k-1}^{l'_3+1,j} = 0$ and $\mathbf{A}_{k-1}^{i,l'_3+1} = 0$. Moreover, similar to the derivation of Equation (65), we have the recursion of the matrix \mathbf{E}_k as

$$\begin{aligned} \mathbf{E}_k^{ij} &= \mathbf{E}_{k-1}^{i+1,j+1} + \mathbf{C}_{k-1}^{i+1,j+1} + \mathbf{B}_{k-1}^{i+1,j+1} \\ &\quad - (\mathbf{E}_{k-1}^{i+1,1} + \mathbf{B}_{k-1}^{i+1,1} + \mathbf{C}_{k-1}^{i+1,1}) \\ &\quad \cdot (\mathbf{E}_{k-1}^{11} + \mathbf{B}_{k-1}^{11} + \mathbf{C}_{k-1}^{11})^{-1} \\ &\quad \cdot (\mathbf{E}_{k-1}^{1,j+1} + \mathbf{B}_{k-1}^{1,j+1} + \mathbf{C}_{k-1}^{1,j+1}). \end{aligned} \tag{74}$$

Based on (61), (65), (67), (70), (72) and (74), we have completed the proof of the Theorem 3.1.

6.2 proof of Corollary 3.3

In case of $l_1 = 0, l_2 = 0, l_4 = 0, l_3 = l$, by (56)-(57), Equation (8) can be written as

$$p_{k+1} =$$

$$\begin{cases} p_k p(\mathbf{x}_{k+1} | \mathbf{x}_k) p(\mathbf{z}_{k+1} | \mathbf{x}_{k+1}), & \text{for } l = 1 \\ & \text{or } l = 0, \\ p_k p(\mathbf{x}_{k+1} | \mathbf{x}_k) p(\mathbf{z}_{k+1} | \mathbf{x}_{k+1}, \mathbf{x}_k) & \text{for } l = 2, \\ p_k p(\mathbf{x}_{k+1} | \mathbf{x}_k) p(\mathbf{z}_{k+1} | \mathbf{x}_{k+1}, \dots, \mathbf{x}_{k-l+2}) & \text{for } l > 2 \end{cases}$$

Thus, we can immediately obtain the recursion (29) by Theorem 3.1. The recursion of \mathbf{E}_k can be written as the three cases of (30), (31) and (32), respectively. At the same time, the matrices \mathbf{B}_k , \mathbf{C}_k , and \mathbf{D}_k become correspondingly appropriate forms.

6.3 proof of Corollary 3.4

In case of $l_1 = 0, l_2 = l, l_3 = 0, l_4 = 0$, i.e., $l'_3 < l'_2 + 1$, by (56)-(57), Equation (8) can be written as

$$p_{k+1} = p_k p(\mathbf{x}_{k+1} | \mathbf{x}_k, \dots, \mathbf{x}_{k-l'_2+1}) p(\mathbf{z}_{k+1} | \mathbf{x}_{k+1}).$$

Thus, we can get (33)-(34) by Theorem 3.1. At the same time, the matrices \mathbf{B}_k , \mathbf{C}_k , and \mathbf{D}_k become correspondingly appropriate forms.

6.4 proof of Corollary 3.5

In case of $l_1 = l$, $l_2 = 0$, $l_3 = 0$, $l_4 = 0$, i.e., $l'_3 < l'_2 + 1$, by (57), Equation (8) can be simplified as

$$p_{k+1} = \begin{cases} p_k p(\mathbf{x}_{k+1}|\mathbf{x}_k) p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}) & \text{for } l = 0, \\ p_k p(\mathbf{x}_{k+1}|\mathbf{x}_k) p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}, \\ \quad \mathbf{z}_k, \dots, \mathbf{z}_{k-l+1}) & \text{for } l > 0. \end{cases}$$

Thus, we can get (40) by Theorem 3.1. At the same time, the matrices \mathbf{B}_k , \mathbf{C}_k , and \mathbf{D}_k become correspondingly appropriate forms.

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